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Discrete Deterministic Chaos

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Discrete Deterministic Chaos

by

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Abstract

Discrete Deterministic Chaos

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The University of Texas at Austin, 2010

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In the course Discrete Deterministic Chaos, Dr. Mark Daniels introduces students to Chaos Theory and explores many topics within the field. Students prove many of the key results that are discussed in class and work through examples of each topic. Connections to the secondary mathematics curriculum are made throughout the course, and students discuss how the topics in the course could be implemented in the classroom. This paper will provide an overview of the topics covered in the course, Discrete Deterministic Chaos, and provide additional discussion on various related topics.

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Introduction

Discrete Deterministic Chaos is the study of iterative systems that have determined outcomes and are dependent on the initial input conditions. These systems may appear to be random and, in many cases, it is difficult to visually distinguish between chaotic and random systems. One characteristic of chaotic systems is that small changes in initial conditions can have dramatic effects on the outcome of the system. A popular example of this sensitivity to initial conditions is the butterfly effect, which states that a butterfly flapping its wings can create a storm on the other side of the world. Many systems encountered in the world have been determined to be chaotic, and many natural systems display self-similarity that can be seen in fractals.

In his course on Discrete Deterministic Chaos, Daniels uses inquiry based learning to expose students to a number of topics in the field [**2**]. The ideas of iteration, fixed points and orbits are explored through Newton's method and the Logistic map. Daniels then uses the ideas of transformations and complex numbers to begin an investigation of fractals. Through the use of in class activities, Daniels creates connections with many secondary mathematics courses and demonstrates how the topics from this course can be implemented in a secondary setting.

Chapter 1: The basics of chaos

Discrete deterministic systems are based on a sequence of values that is created using a defined function under iteration. The sequence is created using an initial value, or seed, x_o and a function. Terms in the sequence are given by:

$$x_o, x_1 = f(x_o), x_2 = f(x_1) = f(f(x_o)) = f^2(x_o), \ldots$$

The sequence that any seed will follow for a given function is called an *orbit*, and these orbits can approach a fixed value, cycle between a set of values, diverge, or behave chaotically, appearing to move at random. A *fixed point* of a function is defined as a point that satisfies the equation $f(x_o) = x_o$. Points can also be eventually fixed if $\forall i \geq n, f(x_i) = x_i$. *Attracting fixed points* of $f(x)$ are fixed points, x_o, such that $|f'(x_o)| < 1$, and *repelling fixed points* of $f(x)$ are fixed points, x_o, such that $|f'(x_o)| > 1$. A fixed point with $|f'(x_o)| = 1$ is a *neutral fixed point*. Daniels [**2**] offers the following theorem, which students proved in class, about attracting fixed points:

> **Attracting Fixed Point Theorem.** Suppose x_o is an attracting fixed point for *F*. Then there is an interval *I* that contains x_o in its interior and in which the following condition is satisfied: if $x \in I$, then $F^n(x) \in I$ for all *n* and, moreover, $F^n(x) \to x_o$ as $n \to \infty$.

A similar theorem holds for repelling fixed points. Being able to identify attracting and repelling fixed points is useful in chaos because they can help identify the orbits of different seed values for an iterative function. It can easily be shown that for any continuous function $f: I \to I$, *f* will have at least one fixed point, and if $|f'(x)| < 1$ for all

x in I, then f will have a unique fixed point in I [**2**]. These statements were also proved in class as an exercise.

Daniels then introduces the concept of hyperbolic points, offering the definition "Fixed points whose derivatives are not equal to one in absolute value are called *hyperbolic fixed points*" [**2**]. Daniels also defines a *hyperbolic periodic point*, "Let p be a periodic point of prime period n. The point p is hyperbolic if $\left|(f^n)'(p)\right| \neq 1$" [**2**]. As an exercise in class, students explore the hyperbolicity of multiple functions in order to determine if fixed points and periodic points are hyperbolic.

Many theorems and results in chaotic systems are based on the derivative of iterative systems, which require the use of the chain rule since iterative systems are compositions of the iterative function. As seen in the definition of a hyperbolic periodic point, in order to determine if a periodic point is hyperbolic or not, the derivative of the nth iteration must be found. As an exercise completed in class, students derive a formula for the $(F^n)'(x_o)$. In order to determine this derivative, we must first suppose that F has a prime period n and that $x_i = F^i(x_0)$. This derivative can be found by first determining that

$$\begin{aligned}(F^2)'(x_0) &= F'(F(x_0)) \cdot F'(x_0) \\ &= F'(x_1) \cdot F'(x_0).\end{aligned}$$

Using induction, if

$$(F^n)'(x_0) = F'(x_{n-1}) \cdot F'(x_{n-2}) \cdot \ldots \cdot F'(x_1) \cdot F'(x_0)$$

then

$$(F^{n+1})'(x_0) = (F(F^n))'(x_0) = F'(F^n(x_0)) \cdot (F^n)'(x_0),$$

thus,

$$(F^{n+1})'(x_0) = F'(x_n) \cdot F'(x_{n-1}) \cdot F'(x_{n-2}) \cdot \ldots \cdot F'(x_1) \cdot F'(x_0).$$

Therefore, the derivative of the *n*th iteration of a function at x_0 is the product of the derivative of the function at the first *n* iterations.

Chapter 2: Newton's Method

The idea of finding roots to a polynomial function is a useful mathematical tool that has a number of applications. While finding roots to linear equations is trivial and finding roots to quadratics can easily be found using multiple techniques, finding roots of higher order polynomials can be difficult or impossible using algebraic techniques.

The Babylonians developed techniques to find square roots, and Daniels uses this approach to introduce the idea of iteration [**2**]. An example is provided to help determine a numerical approximation for $\sqrt{5}$. An initial guess, x_o, can be made for the approximation, and the value of x_o must either be larger or smaller then $\sqrt{5}$. In the case where $x_o < \sqrt{5}$, it can be determined that

$$\begin{aligned} x_o &< \sqrt{5} \\ \sqrt{5}x_o &< 5 \\ x_o &< \sqrt{5} < \frac{5}{x_o} \end{aligned} \tag{1.1}$$

Starting with $\sqrt{5} < x_o$ will also lead to equation (1.1), thus, $\sqrt{5}$ is in between x_o and $\frac{5}{x_o}$. Averaging x_o and $\frac{5}{x_o}$ will lead to a closer approximation of $\sqrt{5}$, so an iterative process can be defined to approximate the root, namely

$$x_n = \frac{1}{2}(x_{n-1} + \frac{5}{x_{n-1}}) \tag{1.2}$$

Therefore, as $n \to \infty$, $x_n \to \sqrt{5}$. This approach, used by the Babylonians, can be useful for determining roots of simple quadratic equations in the form $x^2 - c = 0$ where $c \geq 0$, but will not be useful for solving more complicated quadratic equations or for solving polynomials of higher order.

Newton's method provides a technique that can be used to find the roots of many polynomial functions and greatly expands upon the range of functions for which the Babylonians could approximate solutions. Newton's method consists of an iterative process that is studied by many first year Calculus students. While these students use iterative techniques to calculate zeros using Newton's method, many do not explore the chaotic behavior that certain initial points can lead to. Daniels introduces Newton's Method the first day of class to facilitate the discussion of iteration and later returns to this method to address the behavior of orbits that display different behavior [**2**].

Newton's method uses the equation of a tangent line to a polynomial at a point to approximate the roots of the polynomial. For Newton's method, the iterative process that is used is

$$N(x) = x - \frac{f(x)}{f'(x)}.$$

Daniels uses Newton's method to expand upon the idea of attracting and repelling fixed points [**2**]. By exploring the orbits of selected seeds for certain functions, students are able to identify cases in which Newton's method will be successful and when Newton's method will diverge.

One downside to this traditional form of Newton's method is that there are certain roots that cannot be found based on the behavior of the function. For example, using Newton's method to find any root that is also a vertical tangent will lead to a divergent case. Hetzler [**3**] proposes a continuous case of Newton's method to deal with many of the cases where the traditional version of Newton's method cannot be used to find a root.

Hetzler identifies that the problem which Newton's method encounters in many instances is that x_{n+1} is further from the root then x_n which will lead to the divergence of the iterative sequence. In order to deal with this problem, Hetzler suggests a modification to Newton's method,

$$x(t+h) = x(t) - h\frac{f(x(t))}{f'(x(t))} \tag{1.3}$$

where h is a small, positive constant[**3**, p. 348]. This decreases the size of $f(x_n)/f'(x_n)$, and if an appropriate h is chosen, one can force x_{n+1} to be closer to the root of $f(x)$ then x_n. Hetzler shows that rewriting equation (1.3) in the form

$$\frac{x(t+h)-x(t)}{h} = -\frac{f(x(t))}{f'(x(t))} \tag{1.4}$$

and taking the limit of both sides as h approaches 0 will result in an initial value problem [**2**, p. 349]

$$\frac{dx}{dt} = -\frac{f(x)}{f'(x)},\ x(0) = x_o\,.$$

Hetzler shows that this initial value problem can be solved using Euler's method to arrive at a sequence generated by

$$x_{n+1} = x_n - h[f(x_n)/f'(x_n)]$$

that will approach the root of a polynomial.

Hetzler provides an example of when this continuous case of Newton's method will find roots when the traditional method will not [**3**, p. 350]. The function $f(x) = x^{\frac{1}{3}}$ has only one root, $x = 0$. Using Newton's method, any sequence whose seed is not $x = 0$ will diverge. Using the continuous form of Newton's method, $f'(x) = \frac{1}{3x^{2/3}}$ so $\frac{dx}{dt} = -3x$ and $x(0) = x_o$. The solution to this initial value problem can be solved using separation of variables to yield $x(t) = x_0 e^{-3t}$. This function will approach 0 for any seed x_o as t approaches infinity. Hetzler also provides additional examples where the Newton's method will fail but the continuous version will accurately approximate a root.

Chapter 3: The Logistic

The Logistic model, $F_r(x) = rx(1-x)$, provides a number of cases in which interesting behavior can be observed. Daniels uses the logistic map to explore an iterative function that has chaotic behavior and demonstrates its applications in the real world [**2**]. The fixed points of the logistic map can be determined using the equation

$$rx(1-x) = x$$
$$x(r-1-rx) = 0$$
$$x = 0, \frac{r-1}{r}.$$

For $1 \le r \le 4$, *f* maps [0,1] to [0,1]. There are many interesting behaviors that can be observed in the orbits for different values of r. It can be determined numerically that for values of r between 1 and 3, any seed $x_o \in [0,1]$ will eventually reach a fixed value of $\frac{r-1}{r}$. There are also values of *r* which can lead to a two cycle. In order to determine where a two cycle will occur, we must find an *r* such that $F_r^2(x) = x$ or

$$F_r^2(x) - x = 0. \tag{2.1}$$

By definition,

$$F_r^2(x) = rF_r(x)(1 - F_r(x)). \tag{2.2}$$

Substituting $F_r(x) = rx(1-x)$ into (2.2) yields,

$$F_r^2(x) = r(rx(1-x))(1-(rx(1-x)))$$
$$F_r^2(x) = r^2x - (r^2 + r^3)x^2 + 2r^3x^3 - r^3x^4 \tag{2.3}$$

Using (2.3) to solve (2.1) yields,

$$r^2x - (r^2 + r^3)x^2 + 2r^3x^3 - r^3x^4 = 0 \tag{2.4}$$

The two trivial solutions to this quartic are $x = 0$ and $x = 1$ which are also fixed points of $F_r(x)$. Thus, $F_r(x) - x$ is a factor of $F_r^2(x) - x$. Dividing $F_r^2(x) - x$ by $F_r(x) - x$ will result in

$$-r^3x^3 + (r^3 + r^2)x + (-r^2 - r).$$

Using the quadratic equation, we get

$$x = \frac{(r+1) \pm \sqrt{(r-3)(r+1)}}{2r}.$$

Thus, equation (2.1) will have two real roots whenever $r \geq 3$ which will lead to a two-cycle of the logistic. As seen in Figure 1, the logistic will undergo more bifurcations that will lead to larger cycles as *r* increases, eventually leading to chaotic behavior.

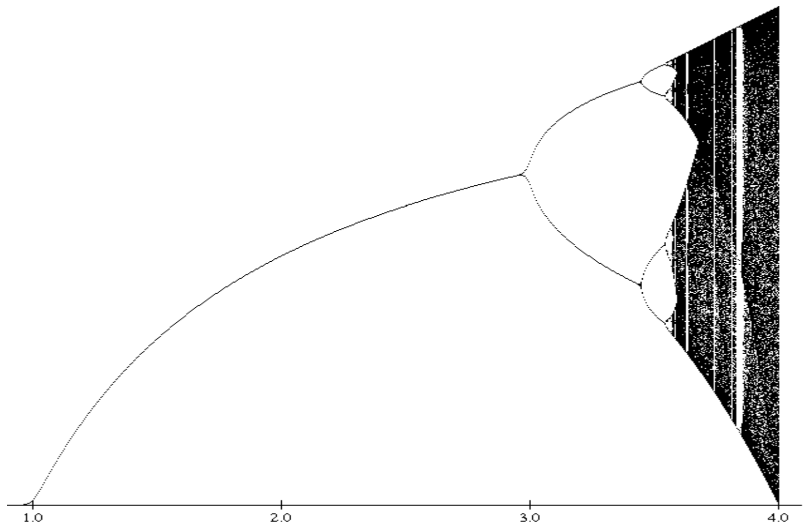

Figure 1 Logistic map Bifurcation from $r = 1$ to $r = 4$

Closer inspection of the bifurcation graph after $r = 3.4$ in Figure 2 reveals that there are many locations where the logistic is chaotic as well as many locations where cycles exist.

Saha and Strogatz [**5**] provide an elegant proof that at $r = 1 + 2\sqrt{2} \approx 3.828$ a 3-period cycle begins, but once again quickly degenerates to chaotic behavior. This 3-cycle can be seen in Figure 2, occurring just before $r = 3.84$ .In order to show that this is where a 3-period cycle begins, Saha starts with four equations:

$$y = rx(1 - x) = f(x)$$
$$z = ry(1 - y) = f^2(x)$$
$$x = rz(1 - z) = f^3(x)$$
$$\frac{d(f^3(x))}{dx} = r^3(1 - 2z)(1 - 2y)(1 - 2x) = 1$$

The first three equations are based on the fact that a three cycle exists and the fourth equation is based on the fact that the three cycle begins with a tangent bifurcation [**5**, p. 45]. Saha uses two changes of variable in order to solve these four equations for the four unknowns and arrives at the conclusion that this tangent bifurcation, and thus the three cycle, begins at $r = 1 + 2\sqrt{2}$ [**5**, pp. 45-47].

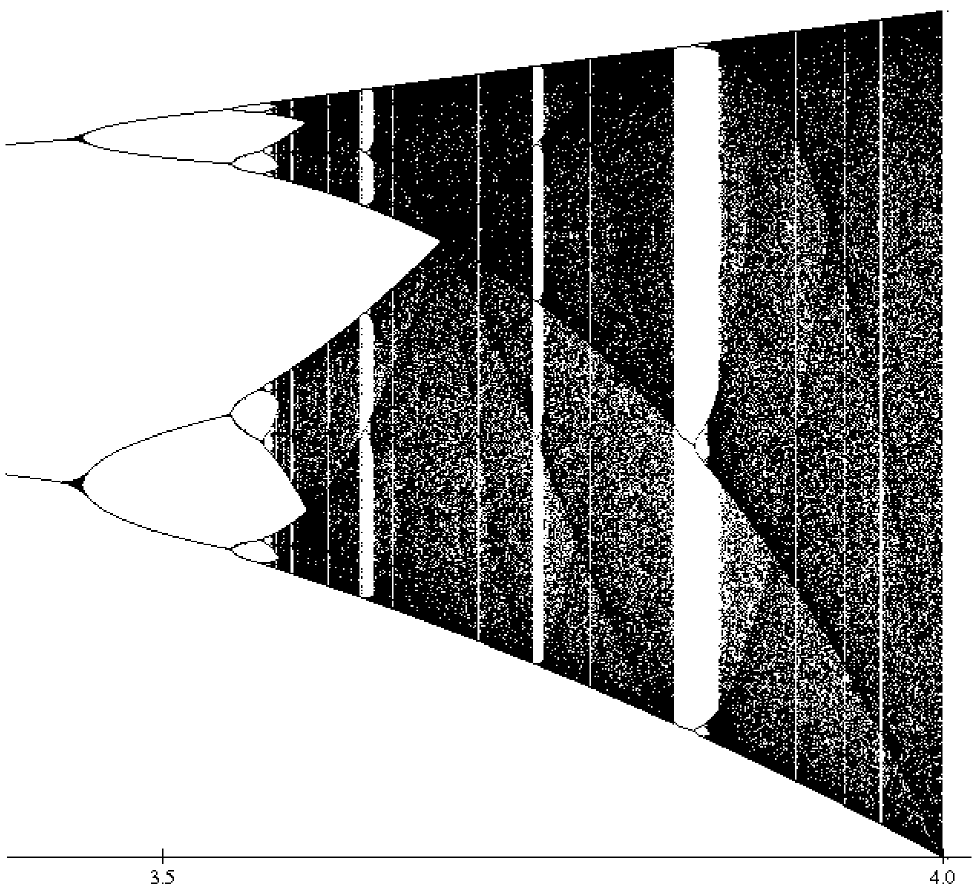

Figure 2 Logistic Bifurcation Map from $r = 3.42$ to $r = 4$

As an end of course project, Daniels places students into groups and assigns each group an article to present to the class. The article presented by the author discussed the behavior of the logistic map after $r = 4$, which is chaotic. Kraft proves that for $r > 4$ the logistic map is a Cantor set, which is chaotic [**4**]. Kraft defines a set under backward iteration,

$$\Lambda_\mu \equiv \bigcap_{n=1}^{\infty} f_\mu^{-n}([0,1])$$

and sets a goal to prove the theorem: "If $\mu > 4$, then Λ_μ is a Cantor set, and the restriction of f_μ to Λ_μ is chaotic" [**4**, p. 400]. According to Kraft, the proof of this theorem is similar to a well published proof that "If $\mu > 2+\sqrt{5}$, then Λ_μ is a Cantor set" [**4**, p. 402]; the difficulty of the proof is that one must show that Λ_μ is a hyperbolic set when $\mu > 4$. Kraft states and proves a variety of lemmas that allow him to prove this theorem, and once he reaches this result, it can be shown that Λ_μ is a Cantor set when $\mu > 4$ [**4**, p. 402]. In order to prove that Λ_μ is a Cantor set, first assume that it is not. Assume that Λ_μ contains an interval, thus $[a,b] \subset \Lambda_\mu$. Since f_μ is differentiable, by the Mean Value Theorem, there exists at least one $c_n \in (a,b)$ such that

$$\frac{f_\mu^n(b) - f_\mu^n(a)}{b-a} = (f_\mu^n)'(c_n)$$

or

$$f_\mu^n(b) - f_\mu^n(a) = (f_\mu^n)'(c_n)(b-a) \tag{2.5}$$

for all $n \geq 1$. Using a lemma and the fact that Λ_μ is hyperbolic, one can show that

$$\left|(f_\mu^n)'(c_n)\right| \geq C\lambda^n \tag{2.6}$$

where $\lambda > 1$. Substituting equation (2.6) into equation (2.5) produces

$$\left| f_\mu^n(b) - f_\mu^n(a) \right| = \left| (f_\mu^n)'(c_n) \right| \cdot \left| b - a \right| \geq C\lambda^n \left| b - a \right| \tag{2.7}$$

Since $\lambda > 1$, for large enough n,

$$\left| f_\mu^n(b) - f_\mu^n(a) \right| > 1 \tag{2.8}$$

but, Λ_μ is invariant so $\{ f_\mu^n(a), f_\mu^n(b) \} \subset \Lambda_\mu \subset [0,1]$ for all n, which is a contradiction. Thus, Λ_μ cannot contain any intervals and is a Cantor set [**4**].

Chapter 4: Fractals

Daniels begins his discussion of fractals with a discussion of Iterated Function Systems, offering a simple definition for a *fractal*, "A subset of R^n which is self similar and whose *Hilbert dimension* exceeds its topological dimension" [**2**]. The function

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \beta \begin{pmatrix} x - x_o \\ y - y_o \end{pmatrix} + \begin{pmatrix} x_o \\ y_o \end{pmatrix}$$

will have a fixed point at $\begin{pmatrix} x_o \\ y_o \end{pmatrix}$ and can be used to define both the Cantor Middle Third Set and the Sierpinski Right Triangle. Once students have examined the fixed points of multiple iterated function systems that use the transformation matrix, Daniels introduces the idea of a rotation matrix,

$$T_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

which will rotate any point or figure around the origin at an angle of α. Composing $A\begin{pmatrix} x \\ y \end{pmatrix}$ and T_α will yield

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \beta T_\alpha \begin{pmatrix} x - x_o \\ y - y_o \end{pmatrix} + \begin{pmatrix} x_o \\ y_o \end{pmatrix}.$$

As an example in class, students create the transformation matrix that will rotate the point p=(2,3) with $\beta = 0.9$, $\alpha = \frac{\pi}{2}$, and $\begin{pmatrix} x_o \\ y_o \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The first iteration of this system will yield the point

$$A\begin{pmatrix} 2 \\ 3 \end{pmatrix} = .9\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2-1 \\ 3-1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.8 \\ 1.9 \end{pmatrix}.$$

Plotting the first 10 points of this iteration will reveal that each successive point is rotated $\frac{\pi}{2}$ radians around the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. A contraction also occurs so that the points spiral to the eventual fixed point of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Daniels then expands upon the transformation function and has students derive a transformation matrix that can be used to translate, rotate and dilate an object in the plane [**2**]. Using properties of matrices, students develop the transformation matrix

$$T = \begin{pmatrix} \kappa\cos\alpha & -\kappa\sin\alpha & g \\ \kappa\sin\alpha & \kappa\cos\alpha & h \\ 0 & 0 & 1 \end{pmatrix}$$

where κ is the dilation factor, α is the angle the object will be rotated in the counterclockwise direction, g is the amount the object will be translated along the x-axis, and h is the amount the object will be translated along the y-axis. Students use T to explore the transformation of multiple objects in the plane and determine the fixed point that these objects will approach under iteration. One example that students examine in class is using the matrix T where $\kappa = .8$, $\alpha = 30°$, g = 5, and h = 3. This will yield the transformation matrix

$$T = \begin{pmatrix} .8\cos 30° & -.8\sin 30° & 5 \\ .8\sin 30° & .8\cos 30° & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

This matrix is used to transform the kite described by the matrix

$$M = \begin{pmatrix} 8 & 10 & 12 & 10 \\ 7 & 2 & 7 & 8 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

where the coordinates of each corner of the kite are the first two elements in each column, and the ones in the bottom row are used in order to create a matrix that can be multiplied by *T*. The first iteration of this system will yield

$$TM \approx \begin{pmatrix} 7.743 & 11.128 & 10.514 & 8.728 \\ 11.049 & 8.386 & 12.649 & 12.543 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Graphing this point as well as the next few iterations will reveal that *M* is being contracted and rotated around a fixed point. Using multiple iterations,

$$T^{100}M = \begin{pmatrix} 1.321 & 1.321 & 1.321 & 1.321 \\ 11.486 & 11.486 & 11.486 & 11.486 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

which shows that *M* will approach the fixed point that is located at approximately, (1.321, 11.486). This fixed point can also be found algebraically by solving the equation

$$T\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Daniels then introduces the Chaos Game and demonstrates how it can be used to generate Sierpinski's Triangle. This exercise shows how what appears to be a random process can generate a complex and chaotic system. Daniels identifies fractals as having the five characteristics:

1. Fractals display detail or fine structure on arbitrarily small scales.
2. Fractals can usually be defined by simple recursive processes.
3. Fractals are too irregular to be described or defined in traditional geometric language.
4. Fractals display some form of self-similarity.
5. Fractals have fractal dimension [**2**].

Daniels proceeds to discuss the properties of a few fractals and has the class explore the area and perimeter of Sierpinski's Triangle and the Koch Snowflake.

The topic of fractal dimension is defined as

$$D = \frac{\log(n)}{\log(\frac{1}{r})}$$

where n is the number of congruent subsets that the fractal can be divided into and r is the ratio of each piece to the length of the original object; this equation was introduced by Hilbert and is referred to as a *Hilbert dimension*.

Another process for measuring fractal dimension the *box count dimension*. This process is useful since it does not require knowing r or n. The process uses a power regression to determine the dimension of a fractal by placing an *r* by *r* grid on the fractal and counting the number of boxes, *n*, of that grid that contain any piece of the fractal. Once multiple (*r*,*n*) pairs have been established, a power regression can be done to solve for the fractal dimension, *D*, in the equation

$$kr^D = n.$$

Once Daniels has introduced the basics of fractals, he proceeds to explore the properties of the Julia and Mandelbrot Sets.

THE QUADRATIC MAP

The quadratic map is given by $Q_c(x) = x^2 + c$. A quick computation will yield that the fixed points of the quadratic map are $x = \pm\sqrt{\frac{1}{4} - c} + \frac{1}{2}$, thus it will have one or two real valued fixed point if $c \le \frac{1}{4}$ and two complex fixed points otherwise. As an exercise in class, students determine that the quadratic map will begin a two cycle after $c = -\frac{3}{4}$ and that when $c < -2$, the quadratic map will be chaotic, similar to the logistic map when $r > 4$ [**2**]. As seen in Figure 5, the quadratic map and logistic map display very similar behavior.

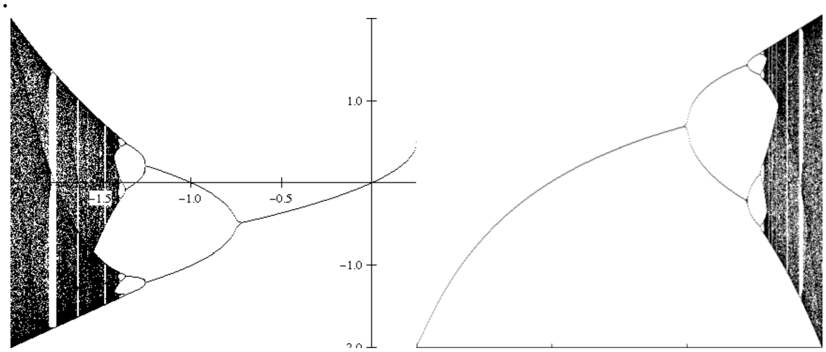

Figure 3: Comparison of Quadratic and Logistic Maps

JULIA SET

The Julia Set is defined using the quadratic function $Q_c(z) = z^2 + c$ where z and c are complex numbers. By fixing the value of c and letting z vary, one can create a Julia Set. In order to determine which points are members of the Julia Set, one must first define the types of behaviors that different seeds will follow. For a given function Q_c, a point z is said to be a *prisoner point* if $\left|Q_c^n(z)\right| \to 0$ as $n \to \infty$. A point z is said to be a *Julia point* if $Q_c^n(z)$ cycles, and a point is said to be an *escaping point* if $\left|Q_c^n(z)\right| \to \infty$ as $n \to \infty$. The Julia Set for a specific value of c is the set of all Julia points for Q_c. Daniels has students explore this behavior for various seeds of the function Q_c, and then has students prove the Escape Criterion Theorem, "Suppose that $|z| \geq |c| > 2$. Then we have $\left|Q_c^n(z)\right| \to \infty$ as $n \to \infty$" [**2**]. The proof of this theorem relies on the use of the triangle inequality. Since

$$\begin{aligned}|a-b| &\geq |a|-|b| \\ |a-(-b)| &\geq |a|-|-b| \\ |a+b| &\geq |a|-|b| \qquad (3.1)\end{aligned}$$

Once equation (3.1) has been obtained from the triangle inequality, one can use this equation to prove the Escape Criterion Theorem:

$$|Q_c(z)| = \left|z^2 + c\right| \geq \left|z^2\right| - |c|. \qquad (3.2)$$

Since $|z| \geq |c|$, equation (3.2) can be rewritten as

$$\begin{aligned}\left|z^2 + c\right| &\geq \left|z^2\right| - |z| \\ &\geq |z|^2 - |z| \\ &\geq |z|(|z|-1) \qquad (3.3)\end{aligned}$$

Because $|z| > 2$, $|z| - 1 > 1$. Let $\lambda > 0$, then $|z| - 1 = 1 + \lambda$. Using this result with equations (3.2) and (3.3),

$$|Q_c(z)| \geq |z|(1 + \lambda)$$

and

$$|Q_c^n(z)| \geq |z|(1 + \lambda)^n$$

Thus, as $n \to \infty$, $|Q_c^n(z)| \to \infty$.

■

A direct result of this proof is that there will be no prisoner points if $|c| > 2$.

MANDELBROT SET

The Mandelbrot set is defined using the quadratic function $Q_c(z) = z^2 + c$ where z is fixed at zero, and complex values of c are input into the function. Daniels [**2**] defines the Mandelbrot set "*M* all *c* vales for which the filled Julia Set is connected. Equivalently,

$$M = \{c \in C \mid \lim_{n \to \infty} Z_n \neq \infty\}$$

where C is the set of complex numbers and:

$$Z_0 = c$$
$$Z_{n+1} = Z_n^2 + c ."$$

Since every point in the Mandelbrot set is the member of a connected Julia set, the Mandelbrot set will include all points where $|c| < 2$, or geometrically, all points in the Mandelbrot set will be within a circle of radius 2 of the origin. The Mandelbrot set can be

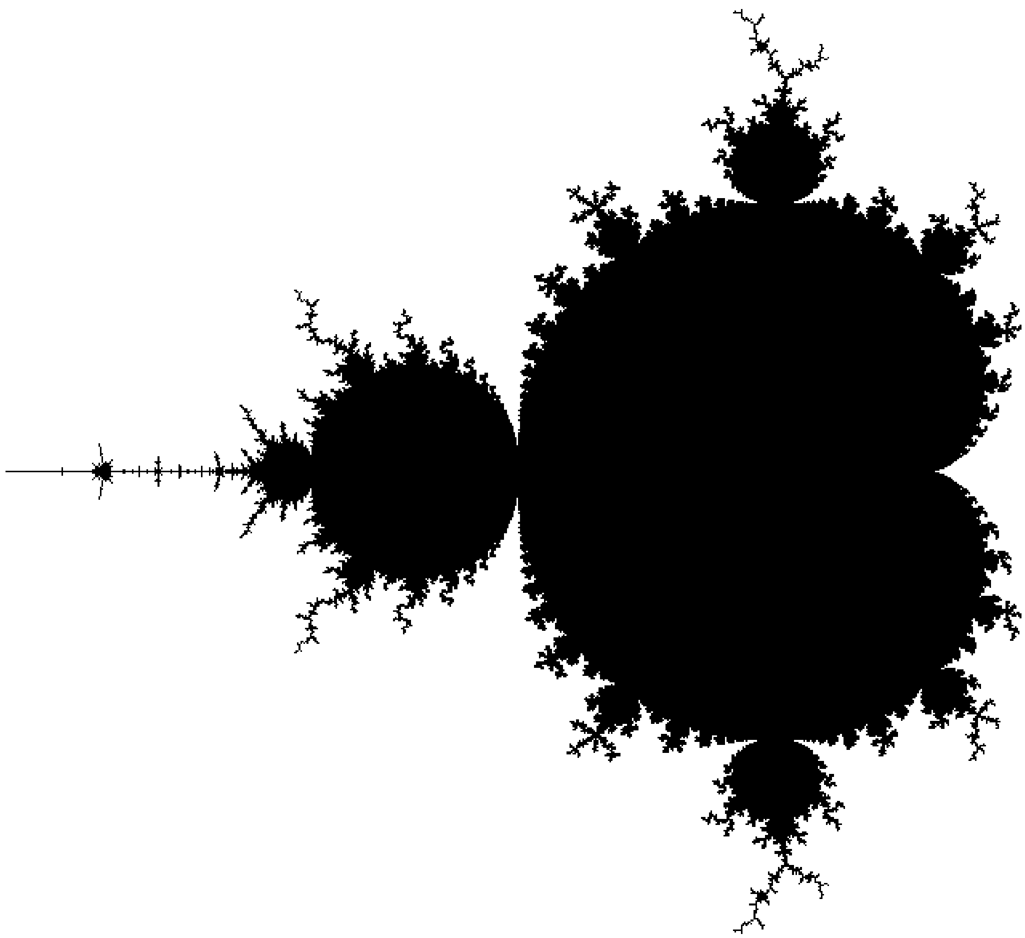

Figure 4 The Mandelbrot Set

graphed multiple ways, with the most basic being to color every point that is in the set black while leaving points not in the set white as seen in Figure 4.

All of the points in the "main cardioid" of the Mandelbrot set will be eventually fixed while seeds in all of the bulbs will cycle. Each bulb of the Mandelbrot set can be assigned a fractional value, and the denominator of the fraction will be the length of the cycle of seed vales that lie in that bulb. The numerator and denominator of the fraction can both be determined by the spokes that radiate from each bulb. The $\frac{2}{5}$ bulb, shown in Figure 5, has five spokes, which is the denominator of the fraction. The numerator of the fraction is determined by the location of the shortest spoke; the number two is assigned to

Figure 5 Two-fifths bulb of the Mandelbrot Set.

the numerator because the shortest spoke is two-fifths of a rotation around the decoration in the counterclockwise direction from the spoke that connects to the set.

If one knows the fraction that is assigned to two bulbs of the set, the fraction of the largest bulb in between these two bulbs can be determined by adding the numerators and denominators of the corresponding bulbs to find the fraction of the new bulb. For example, the $\frac{2}{5}$ bulb is the largest bulb between the $\frac{1}{2}$ and $\frac{1}{3}$ bulbs. Adding the numerators and denominators of the fractions will result in a fraction of $\frac{2}{5}$. Examining the sequence of bulbs that starts with $\frac{1}{2}$ and $\frac{1}{3}$, and continues with this method of addition results in the Fibonacci sequence in both the numerator and denominator of the sequence of fractions,

$$\frac{1}{2},\frac{1}{3},\frac{2}{5},\frac{3}{8},\frac{5}{13},\frac{8}{21},\ldots.$$

This one of the many relationships that Daniels has students explore about the Mandelbrot Set.

Conclusion

Daniels course in Discrete Deterministic Chaos allows students to be introduced to numerous topics in the field of chaos. Many of the topics and activities that students use during the course have direct applications to the secondary classroom. Iteration has many applications in the secondary curriculum that can be incorporated to help deepen students understanding of the material.

The use of iterated function systems could be introduced during a unit in Algebra II on matrices as a geometric application or during a unit on composite functions. This would not only provide students with opportunities to gain experience working with matrices, but would also reinforce ideas of transformations that were learned in Geometry.

Finding roots is a skill that proves to be very useful in a number of mathematical applications. Introducing fixed points to students as a method for finding the roots of a polynomial will allow them to iteratively solve higher order polynomials that they would otherwise not have the ability to factor. Butts provides an innovative way of starting a Calculus course using fixed point iteration, suggesting that using iteration to find the roots of polynomial is a beneficial activity to begin a Calculus course. Butts claims that this approach allows students to be exposed to the idea of a limit and helps stress the importance of derivatives [**1**]. This method could also be introduced in a Pre-Calculus course while studying limits as a means to provide students with additional tools for approximating the roots of polynomials, a useful skill when attempting to graph polynomials.

Newton’s Method, a topic not covered in many secondary Calculus courses, provides students with an application of derivatives that will allow them to enrich their students' ability to find or approximate roots of polynomial functions as well as provide an opportunity to expand upon the ideas of tangent lines. Hetzler's approach to developing a continuous form of Newton's method would be a worthwhile extension of this idea that could be explored while studying separable differential equations. This would allow students to practice their skills in solving a differential equation while also providing an opportunity to review the ideas behind tangent lines and Newton's method. A discussion of Hetzler's technique will also allow for a discussion of when Newton's method will diverge and under which situations one should use the traditional and continuous versions of Newton's method.

An exploration of fractals would provide enrichment opportunities for students in a Geometry class since fractals display many Geometric principles such as self-similarity. Students could explore fractals using computer software or by hand, and those also enrolled in a computer science course could create programs to draw Julia or Mandelbrot sets. Once students are introduced to complex numbers, further exploration of the properties of both the Julia and Mandelbrot set would allow students to gain a deeper understanding of how these sets are generated.

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Vita

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